

Non-linear noise excitation for some space-time fractional stochastic equations in bounded domains

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Abstract

In this paper we study non-linear noise excitation for the following class of space-time fractional stochastic equations in bounded domains:

$$\partial_t^\beta u_t(x) = -\nu(-\Delta)^{\alpha/2} u_t(x) + I_t^{1-\beta} [\lambda \sigma(u) \dot{F}(t, x)]$$

in $(d+1)$ dimensions, where $\nu > 0, \beta \in (0, 1), \alpha \in (0, 2]$. The operator ∂_t^β is the Caputo fractional derivative, $-(-\Delta)^{\alpha/2}$ is the generator of an isotropic stable process and $I_t^{1-\beta}$ is the fractional integral operator. The forcing noise denoted by $\dot{F}(t, x)$ is a Gaussian noise. The multiplicative non-linearity $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be globally Lipschitz continuous. These equations were recently introduced by Mijena and Nane [28]. We first study the existence and uniqueness of the solution of these equations and under suitable conditions on the initial function, we also study the asymptotic behavior of the solution with respect to the parameter λ . In particular, our results are significant extensions of those in [12], [14], [28], and [29].

Keywords: Space-time-fractional stochastic partial differential equations; space-time fractional diffusion in bounded domain; fractional Duhamel's principle; Caputo derivatives; noise excitability.

1 Introduction and statement of the main results.

While fractional calculus has existed in the theoretical realm of mathematics as long as its classical counterpart, the pragmatic applications of said branch of calculus were sparse until the last century, when a rather large number of scientific branches, such as statistical mechanics, theoretical physics, theoretical neuroscience, theory of complex chemical reactions, fluid dynamics, hydrology, and mathematical finance began applying fractional differential equations to problems in said fields; see, for example, Khoshnevisan [21] for an extensive list of references. One such application is, the fractional heat equation $\partial_t^\beta u_t(x) = \Delta u_t(x)$, which describes heat propagation in inhomogeneous media whereas the integer counterpart, the classical heat equation $\partial_t u_t(x) = \Delta u_t(x)$, is used for modeling heat diffusion in homogeneous media. It is well known that when $0 < \beta < 1$, time fractional equations are known to exhibit sub diffusive behavior and are related with anomalous diffusions, or diffusions in non-homogeneous media, with random fractal structures; see, for instance, [26]. Most of the work done so far on the stochastic heat equations have dealt with the usual time derivative, that is $\beta = 1$. But recently, Mijena and Nane have introduced time fractional SPDEs in [28]. These types of time fractional stochastic equations are attractive models that can be used to model phenomenon with random effects with thermal memory. They also studied exponential growth of solutions of time fractional SPDEs–intermittency– under the assumption that the initial function is bounded from below in [29]. In the paper [17] Foondun and Nane have proved asymptotics of the second moment of the solution under various assumptions on the initial function. In addition, a related class of time-fractional SPDE was studied by Karczewska [19], Chen et al. [6], and Baeumer et al [1]. In these papers they proved regularity of the solutions to the time-fractional parabolic type SPDEs using cylindrical Brownian motion in Banach spaces in the sense of [11]. For a comparison of the two approaches to SPDE's see the paper by Dalang and Quer-Sardanyons [10]. The current paper is mainly about a class of space-time fractional stochastic heat equations in bounded domains with Dirichlet boundary conditions.

Next we provide some heuristics before describing our equations. Fix $R > 0$. First let us look at the following space-time fractional equation with Dirichlet boundary conditions (see [9] and [25] for a representation of the solution),

$$\begin{aligned} \partial_t^\beta u_t(x) &= -\nu(-\Delta)^{\alpha/2} u_t(x), \quad x \in B(0, R), t > 0, \\ u_t(x) &= 0, \quad x \in B(0, R)^c, \end{aligned} \tag{1.1}$$

with $\beta \in (0, 1)$ and ∂_t^β is the Caputo fractional derivative which first appeared in [5] and is defined by

$$\partial_t^\beta u_t(x) = \frac{1}{\Gamma(1-\beta)} \int_0^t \partial_r u_r(x) \frac{dr}{(t-r)^\beta}. \tag{1.2}$$

If $u_0(x)$ denotes the initial condition to the above equation, then the solution can be written as

$$u_t(x) = \int_{B(0,R)} G_B(t, x, y) u_0(y) dy.$$

$G_B(t, x, y)$ is the space-time fractional heat kernel. Now consider

$$\partial_t^\beta u_t(x) = -\nu(-\Delta)^{\alpha/2} u_t(x) + f(t, x), \quad (1.3)$$

with the same initial condition $u_0(x)$ and $f(t, x)$ is some nice function. To get the correct version of (1.3) we will make use of **time fractional Duhamel's principle** [33, 35, 34]. Applying the fractional Duhamel principle, the solution to (1.3) is given by

$$u_t(x) = \int_{B(0,R)} G_B(t, x, y) u_0(y) dy + \int_0^t \int_{B(0,R)} G_B(t-r, x, y) \partial_r^{1-\beta} f(r, y) dy dr.$$

Using the definition of the fractional integral

$$I_t^\gamma f(t) := \frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} f(\tau) d\tau,$$

and the property

$$\partial_t^\beta I_t^\beta g(t) = g(t),$$

for every $\beta \in (0, 1)$, and $g \in L^\infty(\mathbb{R}_+)$ or $g \in C(\mathbb{R}_+)$, then by the Duhamel's principle, the mild solution to (1.3) where the force is $f(t, x) = I_t^{1-\beta} g(t, x)$, will be given by

$$u_t(x) = \int_{B(0,R)} G_B(t, x, y) u_0(y) dy + \int_0^t \int_{B(0,R)} G_B(t-r, x, y) g(r, y) dy dr.$$

For more informations on these see [6] and [28].

Our first equation in this paper is the following.

$$\begin{aligned} \partial_t^\beta u_t(x) &= -\mathcal{L}u_t(x) + I_t^{1-\beta} [\lambda \sigma(u_t(x)) \dot{W}(t, x)], \quad x \in B(0, R), \\ u_t(x) &= 0 \quad x \in B(0, R)^c, \end{aligned} \quad (1.4)$$

where \mathcal{L} is the generator of an α -stable process killed upon exiting $B(0, R)$, the initial datum u_0 is a non-random nonnegative measurable function $u_0 : B(0, R) \rightarrow \mathbb{R}_+$ which is strictly positive in a set of positive measure in $B(0, R)$. $\dot{W}(t, x)$ is a space-time white noise with $x \in \mathbb{R}^d$ and $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a globally Lipschitz function with $\sigma(0) = 0$. λ is a positive parameter called the “level of noise”.

We will use an idea in Walsh [36] to make sense of the above equation. Using the above argument, a solution u_t to the above equation will in fact be a solution to the following integral equation.

$$u_t(x) = (\mathcal{G}_B u_0)_t(x) + \lambda \int_0^t \int_{B(0,R)} G_B(t-s, x, y) \sigma(u_s(y)) W(dy ds), \quad (1.5)$$

where

$$(\mathcal{G}_B u_0)_t(x) := \int_{B(0,R)} G_B(t, x, y) u_0(y) dy.$$

Here $G_B(t, x, y)$ denotes the heat kernel of the space-time fractional diffusion equation with Dirichlet boundary conditions in (1.1). In this paper α and β are fixed. We will restrict $\beta \in (0, 1)$ and $\alpha \in (0, 2]$. The relation between the dimension d and the parameters α and β is given by

$$d < (2 \wedge \beta^{-1})\alpha.$$

Observe that when $\beta = 1$, the equation reduces to the well known stochastic heat equation and the above inequality restrict the problem to a one-dimensional one. This is the so called curse of dimensionality explored in [15]. We will need $d < 2\alpha$ to get a finite L^2 -norm of the heat kernel, while $d < \beta^{-1}\alpha$ is needed for an integrability condition needed for ensuring existence and uniqueness of the solution. We will require the following notion of "random-field" solution.

Definition 1.1. A random field $\{u_t(x), t \geq 0, x \in B(0, R)\}$ is called a mild solution of (1.4) if

1. $u_t(x)$ is jointly measurable in $t \geq 0$ and $x \in B(0, R)$;
2. $\forall (t, x) \in [0, \infty) \times \mathbb{R}^d$, $\int_0^t \int_{\mathbb{R}^d} G_B(t-s, x, y) \sigma(u_s(y)) W(dy ds)$ is well-defined in $L^2(\Omega)$; by the Walsh-Dalang isometry this is the same as requiring

$$\sup_{x \in \mathbb{R}^d} \sup_{0 < t \leq T} \mathbb{E}|u_t(x)|^2 < \infty \quad \text{for all } T < \infty.$$

3. The following holds in $L^2(\Omega)$,

$$u_t(x) = (\mathcal{G}_B u_0)_t(x) + \lambda \int_0^t \int_{\mathbb{R}^d} G_B(t-s, x, y) \sigma(u_s(y)) W(dy ds).$$

Before stating our main results, we will mention all the assumptions we need. The first assumption is required for the existence-uniqueness result as well as the upper bound on the second moment of the solution.

Assumption 1.2. • We assume that initial condition is a non-random bounded non-negative function $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}$.

- We assume that $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a globally Lipschitz function satisfying $\sigma(x) \leq L_\sigma |x|$ with L_σ being a positive number.

The following assumption is needed for lower bound on the second moment.

Assumption 1.3. • We will assume that the initial function u_0 is non-negative on a set of positive measure.

- The function σ satisfies $\sigma(x) \geq l_\sigma |x|$ with l_σ being a positive number.

Our first theorem extends the result of Mijena and Nane [28, Theorem 2] for the equation (1.4) in \mathbb{R}^d to the equation with Dirichlet boundary conditions in bounded domains.

Theorem 1.4. *Suppose that $d < (2 \wedge \beta^{-1})\alpha$. Then under Assumption 1.2, there exists a unique random-field solution to (1.4) satisfying*

$$\sup_{x \in B(0, R)} \mathbb{E}|u_t(x)|^2 \leq c_1 e^{c_2 \lambda^{\frac{2\alpha}{\alpha-d\beta}} t} \quad \text{for all } t > 0.$$

Here c_1 and c_2 are positive constants.

Remark 1.5. This theorem says that second moment grows at most exponentially. For small λ , this upper bound is not sharp. But for large λ , this is sharp. Theorem 1.4 implies that a random field solution exists when $d < (2 \wedge \beta^{-1})\alpha$. It follows from this theorem that space-time fractional stochastic equations with space-time white noise is that a random field solution exists in space dimension greater than 1 in some cases, in contrast to the parabolic stochastic heat type equations, the case $\beta = 1$. So in the case $\alpha = 2, \beta < 1/2$, a random field solution exists when $d = 1, 2, 3$. When $\beta = 1$ a random field solution exist only in spatial dimension $d = 1$.

Remark 1.6. Suppose that $d < (2 \wedge \beta^{-1})\alpha$. Using similar ideas in the proof of Theorem 1.4, the results in Theorem 1.4 can be extended to other classes of bounded domains in \mathbb{R}^d . Let D be a bounded domain that is regular as in [9]. Consider the equation

$$\begin{aligned} \partial_t^\beta u_t(x) &= -\mathcal{L}u_t(x) + I_t^{1-\beta}[\lambda\sigma(u_t(x)) \dot{W}(t, x)], \quad t > 0, x \in D, \\ u_t(x) &= 0 \quad x \in D^C, \end{aligned} \tag{1.6}$$

where \mathcal{L} is the generator of an α -stable process killed upon exiting D , the initial datum u_0 is a non-random nonnegative measurable function $u_0 : D \rightarrow \mathbb{R}_+$ which is strictly positive in a set of positive measure in D . $\dot{W}(t, x)$ is a space-time white noise with $x \in \mathbb{R}^d$, and $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a globally Lipschitz function with $\sigma(0) = 0$. Then under Assumption 1.2, there exists a unique random-field solution to (1.6) satisfying

$$\sup_{x \in D} \mathbb{E}|u_t(x)|^2 \leq c_1 e^{c_2 \lambda^{\frac{2\alpha}{\alpha-d\beta}} t} \quad \text{for all } t > 0.$$

Here c_1 and c_2 are positive constants.

Our second theorem is the following.

Theorem 1.7. *Fix $\epsilon > 0$ and let $x \in B(0, R - \epsilon)$, then for any $t > 0$,*

$$\lim_{\lambda \rightarrow \infty} \frac{\log \log \mathbb{E}|u_t(x)|^2}{\log \lambda} = \frac{2\alpha}{\alpha - d\beta},$$

where u_t is the mild solution to (1.4).

Set

$$\mathcal{E}_t(\lambda) := \sqrt{\int_{\mathbb{R}^d} \mathbb{E}|u_t(x)|^2 dx}.$$

and define the nonlinear excitation index by

$$e(t) := \lim_{\lambda \rightarrow \infty} \frac{\log \log \mathcal{E}_t(\lambda)}{\log \lambda}.$$

Corollary 1.8. *The excitation index of the solution to (1.4), $e(t)$ is equal to $\frac{2\alpha}{\alpha-d\beta}$.*

The second class of equation we introduce in this paper is with space colored noise stated as:

$$\begin{aligned} \partial_t^\beta u_t(x) &= \mathcal{L}u_t(x) + I_t^{1-\beta}[\lambda \sigma(u_t(x)) \dot{F}(t, x)], \quad x \in B(0, R), t > 0, \\ u_t(x) &= 0, \quad x \in B(0, R)^c. \end{aligned} \quad (1.7)$$

The only difference with (1.4) is that the noise term is now colored in space. All the other conditions are the same. We now briefly describe the noise. \dot{F} denotes the Gaussian colored noise satisfying the following property,

$$\mathbb{E}[\dot{F}(t, x) \dot{F}(s, y)] = \delta_0(t - s) f(x, y).$$

This can be interpreted more formally as

$$Cov\left(\int \phi dF, \int \psi dF\right) = \int_0^\infty \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \phi_s(x) \psi_s(y) f(x - y), \quad (1.8)$$

where we use the notation $\int \phi dF$ to denote the wiener integral of ϕ with respect to F , and the right-most integral converges absolutely.

We will assume that the spatial correlation of the noise term is given by the following function for $\gamma < d$,

$$f(x, y) := \frac{1}{|x - y|^\gamma}.$$

Following Walsh [36], we define the mild solution of (1.7) as the predictable solution to the following integral equation

$$u_t(x) = (\mathcal{G}_B u_0)_t(x) + \lambda \int_{B(0, R)} \int_0^t G_B(t - s, x, y) \sigma(u_s(y)) F(ds dy). \quad (1.9)$$

As before, we will look at random field solution, which is defined by (1.9). We will also assume the following

$$\gamma < \alpha \wedge d.$$

The condition we should have $\gamma < d$ follows from an integrability condition about the correlation function and $\gamma < \alpha$ comes from an integrability condition needed for the existence and uniqueness of the solution. Our first result on space colored noise case reads as follows.

Theorem 1.9. *Under the Assumption 1.2, there exists a unique random field solution u_t of (1.7) whose second moment satisfies*

$$\sup_{x \in B(0, R)} \mathbb{E}|u_t(x)|^2 \leq c_5 \exp(c_6 \lambda^{2\alpha/(\alpha-\gamma\beta)} t) \quad \text{for all } t > 0.$$

Here the constants c_5, c_6 are positive numbers.

Our main result for the space colored noise equation is the following theorem.

Theorem 1.10. *Fix $t > 0$ and $x \in B(0, R - \epsilon)$, we then have*

$$\lim_{\lambda \rightarrow \infty} \frac{\log \log \mathbb{E}|u_t(x)|^2}{\log \lambda} = \frac{2\alpha}{\alpha - \gamma\beta},$$

where u_t is the unique solution to (1.7).

Corollary 1.11. *The excitation index of the solution to (1.7), $e(t)$ is equal to $\frac{2\alpha}{\alpha - \gamma\beta}$.*

A key difference from the methods used in [12] and [13] is that, here we need to overcome some new technical difficulties. Compared with the usual heat equation with the same boundary conditions, time fractional equations have significantly different behavior. This is the source of the main difficulties we have to overcome. Our method will rely on heat kernel estimates which we will prove later on.

We now briefly give an outline of the paper. In this paper we employ similar methods as in [17] and [12] with crucial changes to prove our main results. We give some preliminary results in section 2. We prove a number of interesting properties of the heat kernel of the time fractional heat type partial differential equations that are essential to the proof of our main results. The most important result in this section is Lemma 2.5. The proofs of the results in the space-time white noise are given in Section 3. In Section 4, we prove the main results about the space colored noise equation. We give an extension of the results stated in the introduction in section 5. Throughout the paper, we use the letter C or c with or without subscripts to denote a constant whose value is not important and may vary from places to places. If $x \in \mathbb{R}^d$, then $|x|$ will denote the euclidean norm of $x \in \mathbb{R}^d$, while when $A \subset \mathbb{R}^d$, $|A|$ will denote the Lebesgue measure of A .

2 Preliminaries.

As mentioned in the introduction, the behaviour of the heat kernel $G_B(t, x)$ will play an important role. This section will mainly be devoted to estimates involving this quantity. Let X_t denote a symmetric α stable process with density function denoted by $p(t, x)$. This is characterized through the Fourier transform which is given by

$$\widehat{p(t, \xi)} = e^{-t\nu|\xi|^\alpha}. \quad (2.1)$$

Let $D = \{D_r, r \geq 0\}$ denote a β -stable subordinator and E_t be its first passage time. It is known that the density of the time changed process X_{E_t} is given by the $G_t(x)$. By conditioning, we have

$$G_t(x) = \int_0^\infty p(s, x) f_{E_t}(s) ds, \quad (2.2)$$

where

$$f_{E_t}(x) = t\beta^{-1}x^{-1-1/\beta}g_\beta(tx^{-1/\beta}), \quad (2.3)$$

where $g_\beta(\cdot)$ is the density function of D_1 and is infinitely differentiable on the entire real line, with $g_\beta(u) = 0$ for $u \leq 0$. Moreover,

$$g_\beta(u) \sim K(\beta/u)^{(1-\beta/2)/(1-\beta)} \exp\{-|1-\beta|(u/\beta)^{\beta/(\beta-1)}\} \quad \text{as } u \rightarrow 0+, \quad (2.4)$$

and

$$g_\beta(u) \sim \frac{\beta}{\Gamma(1-\beta)} u^{-\beta-1} \quad \text{as } u \rightarrow \infty. \quad (2.5)$$

We will need the following properties of the heat kernel of stable process.

•

$$p(t, x) = t^{-d/\alpha} p(1, t^{-1/\alpha} x).$$

•

$$p(st, x) = s^{-d/\alpha} p(t, s^{-1/\alpha} x).$$

- $p(t, x) \geq p(t, y)$ whenever $|x| \leq |y|$.
- It is well known that the transition density $p(t, x)$ of any strictly stable process satisfies the following

$$c_1 \left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}} \right) \leq p(t, x) \leq c_2 \left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}} \right), \quad (2.6)$$

where c_1 and c_2 are positive constants.

The L^2 -norm of the heat kernel can be calculated as follows.

Lemma 2.1 (Lemma 1 in [28]). *Suppose that $d < 2\alpha$, then*

$$\int_{\mathbb{R}^d} G_t^2(x) dx = C^* t^{-\beta d/\alpha}, \quad (2.7)$$

where the constant C^* is given by

$$C^* = \frac{(\nu)^{-d/\alpha} 2\pi^{d/2}}{\alpha \Gamma(\frac{d}{2})} \frac{1}{(2\pi)^d} \int_0^\infty z^{d/\alpha-1} (E_\beta(-z))^2 dz.$$

Here $E_\beta(x)$ is the Mittag-Leffler function defined by

$$E_\beta(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma(1+\beta k)}. \quad (2.8)$$

Remark 2.2. Let $D \subset \mathbb{R}^d$ be a bounded domain. Let $p_D(t, x, y)$ denote the heat kernel of the equation (1.1) when $\beta = 1$. A well known fact is that

$$p_D(t, x, y) \leq p(s, x, y) \quad \text{for all } x, y \in D, t > 0. \quad (2.9)$$

Using the representation from Meerschaert et al. [9] and [25]

$$G_D(t, x, y) = \int_0^\infty p_D(s, x, y) f_{E_t}(s) ds.$$

and equation (2.9) we get

$$G_D(t, x, y) \leq G(t, x, y) \quad \text{for all } x, y \in D, t > 0. \quad (2.10)$$

This fact will be crucial in proving the existence and uniqueness of solutions to equations (1.4) and (1.7).

The next result is crucial in getting the upper bounds for the spatially colored noise equation.

Lemma 2.3 (Lemma 2.7 in [17]). *Suppose that $\gamma < \alpha$, then there exists a constant c_1 such that for all $x, y \in \mathbb{R}^d$, we have*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_t(x - w) G_t(y - z) f(z, w) dw dz \leq \frac{c_1}{t^{\gamma\beta/\alpha}}.$$

The next lemma follows from the previous lemma by using (2.10).

Lemma 2.4. *Let $D \subset \mathbb{R}^d$ be a bounded domain. Suppose that $\gamma < \alpha$, then there exists a constant c_1 such that for all $x, y \in D$, we have*

$$\int_D \int_D G_D(t, x, w) G_D(t, y, z) f(z, w) dw dz \leq \frac{c_1}{t^{\gamma\beta/\alpha}}.$$

The next proposition is the crucial result in proving the lower bounds in Theorems 1.4 and 1.10.

Proposition 2.5. *Fix $\epsilon > 0$, then there exists $t_0 > 0$ such that for all $x, y \in B(0, R - \epsilon)$ and for all $t < t_0$ and $|x - y| < t^{\beta/\alpha}$ we have*

$$G_B(t, x, y) \geq Ct^{-\beta d/\alpha},$$

for some constant $C > 0$.

Proof. We use the representation

$$G_B(t, x, y) = \int_0^\infty p_B(s, x, y) f_{E_t}(s) ds.$$

By proposition 2.1 in [12] there exists a $T_0 > 0$ such that $p_D(t, x, y) \geq c_1 p(t, x, y)$ whenever $t \leq T_0$. Now this and the representation (2.3) with a change of variables we get

$$G_B(t, x, y) \geq \int_0^{T_0} p_B(s, x, y) f_{E_t}(s) ds$$

$$\begin{aligned}
&\geq c_1 \int_0^{T_0} p(s, x, y) f_{E_t}(s) ds \\
&= c_1 \int_{tT_0^{-1/\beta}}^\infty p((t/u)^\beta, x, y) g_\beta(u) du. \tag{2.11}
\end{aligned}$$

Now suppose that $tT_0^{-1/\beta} < 1/2$ and $|x-y| < t^{\beta/\alpha}$ hence $t/|x-y|^{\alpha/\beta} > 1$ and for $u < t/|x-y|^{\alpha/\beta}$ we have $(t/u)^\beta > |x-y|^\alpha$ or equivalently $|x-y| < [(t/u)^\beta]^{1/\alpha}$, therefore using all of these observations with (2.6) we obtain

$$\begin{aligned}
G_B(t, x, y) &\geq c_1 \int_{tT_0^{-1/\beta}}^\infty p((t/u)^\beta, x, y) g_\beta(u) du \\
&\geq c_1 \int_{tT_0^{-1/\beta}}^{t/|x-y|^{\alpha/\beta}} p((t/u)^\beta, x, y) g_\beta(u) du \\
&\geq C \int_{tT_0^{-1/\beta}}^{t/|x-y|^{\alpha/\beta}} (1/(t/u)^\beta)^{d/\alpha} g_\beta(u) du \\
&\geq C \int_{1/2}^1 t^{-\beta d/\alpha} u^{\beta d/\alpha} g_\beta(u) du \\
&= Ct^{-\beta d/\alpha} \int_{1/2}^1 u^{\beta d/\alpha} g_\beta(u) du = Ct^{-\beta d/\alpha}.
\end{aligned}$$

□

Remark 2.6. Recall that for any $t > 0$ and $x \in B(0, R)$

$$(\mathcal{G}_B u)_t(x) := \int_{B(0, R)} G_B(t, x, y) u_0(y) dy.$$

By remark 2.2 in Foondun et al. [12] we know that for fixed $\epsilon > 0$ we have

$$h_t := \inf_{x \in B(0, R-\epsilon)} \inf_{s \leq t} (\tilde{\mathcal{G}}_B u)_s(x) = \inf_{x \in B(0, R-\epsilon)} (\tilde{\mathcal{G}}_D u)_t(x) > 0,$$

where $(\tilde{\mathcal{G}}_B u)_s(x) = \int_{B(0, R)} p_B(t, x, y) u_0(y) dy$ is the killed semigroup of stable process and is the solution of (1.1) when $\beta = 1$.

By a simple conditioning we have

$$\begin{aligned}
(\mathcal{G}_B u)_{s+t_0}(x) &:= \int_{B(0, R)} G_B(s+t_0, x, y) u_0(y) dy = \int_0^\infty (\tilde{\mathcal{G}}_B u)_{s'}(x) f_{E_{s+t_0}}(s') ds' \\
&\geq \int_0^{s+t_0} (\tilde{\mathcal{G}}_B u)_s(x) f_{E_{s+t_0}}(s) ds \\
&\geq h_{s+t_0} \int_0^{s+t_0} f_{E_{s+t_0}}(s) ds := g_t > 0,
\end{aligned} \tag{2.12}$$

for any fixed $t_0 > 0$ and all $s \leq t$.

We end this section with a few results from [12] and [13]. These will be useful for the proofs of our main results.

Proposition 2.7. *[Proposition 2.5 in [13]] Let $\rho > 0$ and suppose $f(t)$ is a locally integrable function satisfying*

$$f(t) \leq c_1 + \kappa \int_0^t (t-s)^{\rho-1} f(s) ds \quad \text{for all } t > 0,$$

where c_1 is some positive number. Then, we have

$$f(t) \leq c_2 \exp(c_3(\Gamma(\rho))^{1/\rho} \kappa^{1/\rho} t) \quad \text{for all } t > 0,$$

for some positive constants c_2 and c_3 .

Also we give the following converse.

Proposition 2.8 (Proposition 2.6 in [13]). *Let $\rho > 0$ and suppose $f(t)$ is nonnegative, locally integrable function satisfying*

$$f(t) \geq c_1 + \kappa \int_0^t (t-s)^{\rho-1} f(s) ds \quad \text{for all } t > 0,$$

where c_1 is some positive number. Then, we have

$$f(t) \geq c_2 \exp(c_3(\Gamma(\rho))^{1/\rho} \kappa^{1/\rho} t) \quad \text{for all } t > 0,$$

for some positive constants c_2 and c_3 .

Proposition 2.9 (Proposition 2.6 in [12]). *Let $T < \infty$ and $\eta > 0$. Suppose that $f(t)$ is a positive locally integrable function satisfying*

$$f(t) \geq c_2 + \kappa \int_0^t (t-s)^{\eta-1} f(s) ds \quad \text{for all } 0 \leq t \leq T, \quad (2.13)$$

where c_2 is some positive number. Then for any $t \in (0, T]$, we have the following

$$\liminf_{\kappa \rightarrow \infty} \frac{\log \log f(t)}{\log \lambda} \geq \frac{1}{\eta}.$$

Lemma 2.10 (Lemma 2.4 in [12]). *Let $\rho > 0$ and $S(t) = \sum_{k=1}^{\infty} \left(\frac{t}{k^\rho}\right)^k$. For any fixed $t > 0$, we have*

$$\liminf_{\theta \rightarrow \infty} \frac{\log \log S(\theta t)}{\log \theta} \geq \frac{1}{\rho}.$$

3 Proofs for the white noise case.

3.1 Proofs of Theorem 1.4.

Proof. The proof follows main steps in [12] and [17] with some crucial changes. We first show the existence of a unique solution. This follows from a standard

Picard iteration; see [36], so we just briefly spell out the main ideas. For more information, see [28]. Set

$$u_t^{(0)}(x) := (\mathcal{G}_B u_0)_t(x)$$

and

$$u_t^{(n+1)}(x) := (\mathcal{G}_B u_0)_t(x) + \lambda \int_0^t \int_{B(0,R)} G_B(t-s, x, y) \sigma(u_s^{(n)}(y)) W(dy ds) \quad \text{for } n \geq 0.$$

Define $D_n(t, x) := \mathbb{E}|u_t^{(n+1)}(x) - u_t^{(n)}(x)|^2$ and $H_n(t) := \sup_{x \in \mathbb{R}^d} D_n(t, x)$. We will prove the result for $t \in [0, T]$, where T is some fixed number. We now use this notation, (2.10), together with Walsh's isometry and the assumption on σ to write

$$\begin{aligned} D_n(t, x) &= \lambda^2 \int_0^t \int_{B(0,R)} G_B^2(t-s, x, y) \mathbb{E}|\sigma(u_s^{(n)}(y)) - \sigma(u_s^{(n-1)}(y))|^2 dy ds \\ &\leq \lambda^2 L_\sigma^2 \int_0^t H_{n-1}(s) \int_{\mathbb{R}^d} G^2(t-s, x, y) dy ds \\ &\leq \lambda^2 L_\sigma^2 \int_0^T \frac{H_{n-1}(s)}{(t-s)^{d\beta/\alpha}} ds \end{aligned}$$

We therefore have

$$H_n(t) \leq \lambda^2 L_\sigma^2 \int_0^T \frac{H_{n-1}(s)}{(t-s)^{d\beta/\alpha}} ds.$$

We now note that the integral appearing on the right hand side of the above display is finite when $d < \alpha/\beta$. Hence, by Lemma 3.3 in Walsh [36], the series $\sum_{n=0}^\infty H_n^{\frac{1}{2}}(t)$ converges uniformly on $[0, T]$. Therefore, the sequence $\{u_n\}$ converges in L^2 and uniformly on $[0, T] \times \mathbb{R}^d$ and the limit satisfies (1.5). We can prove uniqueness in a similar way. We now turn to the proof of the exponential bound. From Walsh's isometry, we have

$$\mathbb{E}|u_t(x)|^2 = |(\mathcal{G}_B u_0)_t(x)|^2 + \lambda^2 \int_0^t \int_{B(0,R)} G_B^2(t-s, x, y) \mathbb{E}|\sigma(u_s(y))|^2 dy ds.$$

Since we are assuming that the initial function (condition) is bounded, we have that $|(\mathcal{G}_B u_0)_t(x)|^2 \leq c_1$ and by (2.10) the second term is bounded by

$$\begin{aligned} &\lambda^2 L_\sigma^2 \int_0^t \int_{B(0,R)} G_B^2(t-s, x, y) \mathbb{E}|u_s(y)|^2 dy ds \\ &\leq c_1 \lambda^2 L_\sigma^2 \int_0^t \frac{1}{(t-s)^{d\beta/\alpha}} \sup_{y \in B(0,R)} \mathbb{E}|u_s(y)|^2 dy ds. \end{aligned}$$

We therefore have

$$\sup_{x \in B(0,R)} \mathbb{E}|u_s(x)|^2 \leq c_1 + c_2 \lambda^2 L_\sigma^2 \int_0^t \frac{1}{(t-s)^{d\beta/\alpha}} \sup_{y \in B(0,R)} \mathbb{E}|u_s(y)|^2 ds.$$

The renewal inequality in Proposition 2.7 with $\rho = (\alpha - d\beta)/\alpha$ proves the result. \square

3.2 Proof of Theorem 1.7.

Set $\mathcal{S}_t(\lambda) := \sup_{x \in B(0, R)} \mathbb{E}|u_t(x)|^2$. We first state following proposition which follows from Theorem 1.4.

Proposition 3.1. *Fix $t > 0$, then*

$$\limsup_{\lambda \rightarrow \infty} \frac{\log \log \mathcal{S}_t(\lambda)}{\log \lambda} \leq \frac{2\alpha}{\alpha - \beta d}.$$

For any fixed $\epsilon > 0$, set

$$\mathcal{I}_{\epsilon, t}(\lambda) := \inf_{x \in B(0, R-\epsilon)} \mathbb{E}|u_t(x)|^2.$$

Next we give a proposition that gives the lower bound in Theorem 1.7

Proposition 3.2. *For any fixed $\epsilon > 0$, there exists a $t_0 > 0$ such that for all $t \leq t_0$,*

$$\liminf_{\lambda \rightarrow \infty} \frac{\log \log \mathcal{I}_{\epsilon, t}(\lambda)}{\log \lambda} \geq \frac{2\alpha}{\alpha - \beta d}.$$

Proof. The proof of the proposition will rely on the following observation. From Walsh isometry, we have

$$\begin{aligned} \mathbb{E}|u_t(x)|^2 &= |(\mathcal{G}_B u_0)_t(x)|^2 + \lambda^2 \int_0^t \int_{B(0, R)} G_B^2(t-s, x, y) \mathbb{E}|\sigma(u_s(y))|^2 dy ds. \\ &= I_1 + I_2. \end{aligned}$$

We fix $\epsilon > 0$ and choose a t_0 as in Proposition 2.5.

For $x \in B(0, R-\epsilon)$ we have $\mathcal{G}_B(t, x) \geq g_{t_0}$ by Remark 2.6. Hence $I_1 \geq g_{t_0}^2$. We now prove the lower bound for I_2 .

$$\begin{aligned} I_2 &\geq (\lambda l_\sigma)^2 \int_0^t \int_{B(0, R)} G_B^2(t-s, x, y) \mathbb{E}|u_s(y)|^2 dy ds \\ &\geq (\lambda l_\sigma)^2 \int_0^t \mathcal{I}_{\epsilon, s}(\lambda) \int_{B(0, R-\epsilon)} G_B^2(t-s, x, y) dy ds. \end{aligned}$$

Set $A := \{y \in B(0, R-\epsilon) : |x-y| \leq (t-s)^{\beta/\alpha}\}$. Since $t-s \leq t_0$, we have $|A| \geq c_1(t-s)^{d\beta/\alpha}$. Now using Proposition 2.5, we have

$$\begin{aligned} \int_{B(0, R-\epsilon)} G_B^2(t-s, x, y) dy &\geq c_2 \int_A \frac{1}{(t-s)^{2\beta d/\alpha}} \\ &= c_3 \frac{1}{(t-s)^{\beta d/\alpha}}. \end{aligned}$$

We thus have

$$I_2 \geq c_4 \lambda^2 \int_0^t \frac{\mathcal{I}_{\epsilon, s}(\lambda)}{(t-s)^{\beta d/\alpha}} ds.$$

Combining the above estimates we have

$$\mathcal{I}_{\epsilon,t}(\lambda) \geq g_{t_0}^2 + c_4 \lambda^2 \int_0^t \frac{\mathcal{I}_{\epsilon,s}(\lambda)}{(t-s)^{\beta d/\alpha}} ds.$$

We now apply Proposition 2.9. □

Proof of Theorem 1.7. The proof of the result when $t \leq t_0$ follows from the two propositions above. To prove the theorem for all $t > 0$, we only need to prove the above proposition for all $t > 0$. For any fixed $T, t > 0$, by changing variables we have

$$\begin{aligned} \mathbb{E}|u_{t+T}(x)|^2 &\geq |(\mathcal{G}_B u_0)_{t+T}(x)|^2 + \lambda^2 \int_0^{t+T} \int_{B(0,R)} G_B^2(t+T-s, x, y) \mathbb{E}|\sigma(u_s(y))|^2 dy ds \\ &\geq |(\mathcal{G}_B u_0)_{t+T}(x)|^2 + \lambda^2 \int_0^T \int_{B(0,R)} G_B^2(t+T-s, x, y) \mathbb{E}|\sigma(u_s(y))|^2 dy ds \\ &\quad + \lambda^2 \int_0^t \int_{B(0,R)} G_B^2(t+T-s, x, y) \mathbb{E}|\sigma(u_{s+T}(y))|^2 dy ds. \end{aligned}$$

This gives

$$\mathbb{E}|u_{t+T}(x)|^2 \geq |(\mathcal{G}_B u_0)_{t+T}(x)|^2 + \lambda^2 l_\sigma^2 \int_0^t \int_{B(0,R)} G_B^2(t+T-s, x, y) \mathbb{E}|u_{s+T}(y)|^2 dy ds,$$

since $|(\mathcal{G}_B u_0)_{t+T}(x)|^2$ strictly positive, we can use the proof of the above proposition with an obvious modification to conclude that

$$\liminf_{\lambda \rightarrow \infty} \frac{\log \log \mathbb{E}|u_{t+T}(x)|^2}{\log \lambda} \geq \frac{2\alpha}{\alpha - \beta d},$$

for $x \in B(0, R - \epsilon)$ and small t . □

Proof of Corollary 1.8. Note that

$$\int_{B(0,R)} \mathbb{E}|u_t(x)|^2 dx \leq C R^d \sup_{x \in B(0,R)} \mathbb{E}|u_t(x)|^2,$$

and

$$\int_{B(0,R)} \mathbb{E}|u_t(x)|^2 dx \geq C(R - \epsilon)^d \inf_{x \in B(0,R-\epsilon)} \mathbb{E}|u_t(x)|^2.$$

We now apply Theorem 1.7 and use the definition of $\mathcal{E}_t(\lambda)$ to obtain the result. □

4 Proofs for the colored noise case.

4.1 Proof of Theorem 1.9.

Proof. The proof of existence and uniqueness is standard as in [12] and [17]. We give the details for the convenience of the reader. For more information, see [36]. We set

$$u^{(0)}(t, x) := (\mathcal{G}_B u_0)_t(x),$$

and

$$u^{(n+1)}(t, x) := (\mathcal{G}_B u_0)_t(x) + \lambda \int_0^t \int_{B(0,R)} G_B(t-s, x, y) \sigma(u^{(n)}(s, y)) F(dy ds), \quad n \geq 0.$$

Define $D_n(t, x) := \mathbb{E}|u^{(n+1)}(t, x) - u^{(n)}(t, x)|^2$, $H_n(t) := \sup_{x \in \mathbb{R}^d} D_n(t, x)$ and $\Sigma(t, y, n) = |\sigma(u^{(n)}(t, y)) - \sigma(u^{(n-1)}(t, y))|$. We will prove the result for $t \in [0, T]$ where T is some fixed number. We now use this notation together with the covariance formula (1.8) and the assumption on σ to write

$$\begin{aligned} D_n(t, x) &= \lambda^2 \int_0^t \int_{B(0,R)} \int_{B(0,R)} G_B(t-s, x, y) G_B(t-s, x, z) \mathbb{E}[\Sigma(s, y, n) \Sigma(s, z, n)] f(y, z) dy dz ds. \end{aligned}$$

Now we estimate the expectation on the right hand side using Cauchy-Schwartz inequality.

$$\begin{aligned} \mathbb{E}[\Sigma(s, y, n) \Sigma(s, z, n)] &\leq L_\sigma^2 \mathbb{E}|u^{(n)}(s, y) - u^{(n-1)}(s, y)| |u^{(n)}(s, z) - u^{(n-1)}(s, z)| \\ &\leq L_\sigma^2 \left(\mathbb{E}|u^{(n)}(s, y) - u^{(n-1)}(s, y)|^2 \right)^{1/2} \\ &\quad \left(\mathbb{E}|u^{(n)}(s, z) - u^{(n-1)}(s, z)|^2 \right)^{1/2} \\ &\leq L_\sigma^2 \left(D_{n-1}(s, y) D_{n-1}(s, z) \right)^{1/2} \\ &\leq L_\sigma^2 H_{n-1}(s). \end{aligned}$$

Hence we have for $\gamma < \alpha$ using Lemma 2.3

$$\begin{aligned} D_n(t, x) &\leq \lambda^2 L_\sigma^2 \int_0^t H_{n-1}(s) \int_{B(0,R)} \int_{B(0,R)} G_B(t-s, x, y) G_B(t-s, x, z) f(y, z) dy dz ds \\ &\leq \lambda^2 L_\sigma^2 \int_0^t H_{n-1}(s) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_{t-s}(x-y) G_{t-s}(x-z) f(y, z) dy dz ds \\ &\leq c_1 \lambda^2 L_\sigma^2 \int_0^t \frac{H_{n-1}(s)}{(t-s)^{\gamma\beta/\alpha}} ds. \end{aligned}$$

We therefore have

$$H_n(t) \leq c_1 \lambda^2 L_\sigma^2 \int_0^t \frac{H_{n-1}(s)}{(t-s)^{\gamma\beta/\alpha}} ds.$$

We now note that the integral appearing on the right hand side of the above display is finite when $\gamma < \alpha/\beta$. Hence, by Lemma 3.3 in Walsh [36], the series $\sum_{n=0}^\infty H_n^{\frac{1}{2}}(t)$ converges uniformly on $[0, T]$. Therefore, the sequence $\{u_n\}$ converges in L^2 and uniformly on $[0, T] \times \mathbb{R}^d$ and the limit satisfies (1.9). We can prove uniqueness in a similar way.

We now turn to the proof of the exponential bound. Set

$$A(t) := \sup_{x \in \mathbb{R}^d} \mathbb{E}|u_t(x)|^2.$$

We claim that there exist constants c_4, c_5 such that for all $t > 0$, we have

$$A(t) \leq c_4 + c_5 (\lambda L_\sigma)^2 \int_0^t \frac{A(s)}{(t-s)^{\beta\gamma/\alpha}} ds.$$

The renewal inequality in Proposition 2.7 with $\rho = (\alpha - \gamma\beta)/\alpha$ then proves the exponential upper bound. To prove this claim, we start with the mild formulation given by (1.9), then take the second moment to obtain the following

$$\begin{aligned} \mathbb{E}|u_t(x)|^2 &= |(\mathcal{G}_B u)_t(x)|^2 \\ &+ \lambda^2 \int_0^t \int_{B \times B} G_B(t-s, x, y) G_B(t-s, x, z) f(y, z) \mathbb{E}[\sigma(u_s(y))\sigma(u_s(z))] dy dz ds \\ &= I_1 + I_2. \end{aligned} \tag{4.1}$$

Since u_0 is bounded, we have $I_1 \leq c_4$. Next we use the assumption on σ together with Hölder's inequality to see that

$$\begin{aligned} \mathbb{E}[\sigma(u_s(y))\sigma(u_s(z))] &\leq L_\sigma^2 \mathbb{E}[u_s(y)u_s(z)] \\ &\leq L_\sigma^2 [\mathbb{E}|u_s(y)|^2]^{1/2} [\mathbb{E}|u_s(z)|^2]^{1/2} \\ &\leq L_\sigma^2 \sup_{x \in \mathbb{R}^d} \mathbb{E}|u_s(x)|^2. \end{aligned} \tag{4.2}$$

Therefore, using Lemma 2.3 the second term I_2 is thus bounded as follows.

$$I_2 \leq c_5 (\lambda L_\sigma)^2 \int_0^t \frac{A(s)}{(t-s)^{\beta\gamma/\alpha}} ds.$$

Combining the above estimates, we obtain the required result in the claim. \square

4.2 Proof of Theorem 1.10.

The proof is inspired by the methods in [12] and [17]. Set $B = B(0, R)$ and $B_\epsilon = B(0, R - \epsilon)$. We will use the following notation $B^2 = B \times B$ and $B_\epsilon^2 = B_\epsilon \times B_\epsilon$. The starting point of the proof of the lower bound hinges on the following recursive argument.

$$\begin{aligned} \mathbb{E}|u_{t+\tilde{t}}(x)|^2 &= |(\mathcal{G}_B u)_{t+\tilde{t}}(x)|^2 + \lambda^2 \int_0^{t+\tilde{t}} \int_{B^2} \\ &\quad G_B(t+\tilde{t}-s_1, x, z_1) G_B(t+\tilde{t}-s_1, x, z'_1) \mathbb{E}[\sigma(u_{s_1}(z_1)) \sigma(u_{s_1}(z'_1)) f(z_1, z'_1)] dz_1 dz'_1 ds_1. \end{aligned}$$

We now use the assumption that $\sigma(x) \geq l_\sigma |x|$ for all x together with a change of variable to reduce the above to

$$\begin{aligned} \mathbb{E}|u_{t+\tilde{t}}(x)|^2 &\geq |(\mathcal{G}_B u)_{t+\tilde{t}}(x)|^2 + \lambda^2 l_\sigma^2 \int_0^t \int_{B^2} \\ &\quad G_B(t-s_1, x, z_1) G_B(t-s_1, x, z'_1) \mathbb{E}|u_{s_1+\tilde{t}}(z_1) u_{s_1+\tilde{t}}(z'_1)| f(z_1, z'_1) dz_1 dz'_1 ds_1. \end{aligned}$$

We also have

$$\begin{aligned} \mathbb{E}|u_{s_1+\tilde{t}}(z_1) u_{s_1+\tilde{t}}(z'_1)| &\geq |(\mathcal{G}_B u)_{s_1+\tilde{t}}(z_1) (\mathcal{G}_B u)_{s_1+\tilde{t}}(z'_1)| + \lambda^2 l_\sigma^2 \int_0^{s_1} \int_{B^2} \\ &\quad G_B(s_1-s_2, z_1, z_2) G_B(s_1-s_2, z'_1, z'_2) \mathbb{E}|u_{s_2+\tilde{t}}(z_2) u_{s_2+\tilde{t}}(z'_2)| f(z_2, z'_2) dz_2 dz'_2 ds_2. \end{aligned}$$

We set $z_0 = z'_0 := x$ and $s_0 := t$ and continue the recursion as above to obtain

$$\begin{aligned} \mathbb{E}|u_{t+\tilde{t}}(x)|^2 &\geq |(\mathcal{G}_B u)_{t+\tilde{t}}(x)|^2 \\ &\quad + \sum_{k=1}^{\infty} (\lambda l_\sigma)^{2k} \int_0^t \int_{B^2} \int_0^{s_1} \int_{B^2} \cdots \int_0^{s_{k-1}} \int_{B^2} |(\mathcal{G}_B u)_{s_k+\tilde{t}}(z_k) (\mathcal{G}_B u)_{s_k+\tilde{t}}(z'_k)| \\ &\quad \prod_{i=1}^k G_B(s_{i-1}-s_i, z_{i-1}, z_i) G_B(s_{i-1}-s_i, z'_{i-1}, z'_i) f(z_i, z'_i) dz_i dz'_i ds_i. \end{aligned} \tag{4.3}$$

Proposition 4.1. *Fix $\epsilon > 0$. Then for all $x \in B(0, R - \epsilon)$ and $0 \leq t \leq t_0$*

$$\mathbb{E}|u_{t+\tilde{t}}(x)|^2 \geq g_t^2 + g_t^2 \sum_{k=1}^{\infty} (\lambda^2 l_\sigma^2 c_1)^k \left(\frac{t}{k}\right)^{k(\alpha-\gamma\beta)/\alpha},$$

where c_1 is a positive constant depending on α and γ and $\tilde{t} > 0$ is a fixed constant.

Proof. We will look at the following term which comes from the recursive relation described above,

$$\sum_{k=1}^{\infty} (\lambda l_{\sigma})^{2k} \int_0^t \int_{B^2} \int_0^{s_1} \int_{B^2} \cdots \int_0^{s_{k-1}} \int_{B^2} |(\mathcal{G}_B u)_{s_k+\tilde{t}}(z_k)(\mathcal{G}_B u)_{s_k+\tilde{t}}(z'_k)| \\ \prod_{i=1}^k G_B(s_{i-1} - s_i, z_{i-1}, z_i) G_B(s_{i-1} - s_i, z'_{i-1}, z'_i) f(z_i, z'_i) dz_i dz'_i ds_i.$$

Using the fact from Remark 2.6 that for $z_k, z'_k \in B_{\epsilon}$

$$\begin{aligned} (\mathcal{G}_B u)_{s_k+\tilde{t}}(z_k)(\mathcal{G}_B u)_{s_k+\tilde{t}}(z'_k) \\ \geq \inf_{x,y \in B_{\epsilon}} \inf_{0 \leq s \leq t} (\mathcal{G}_B u)_{s+\tilde{t}}(x)(\mathcal{G}_B u)_{s+\tilde{t}}(y) \quad (4.4) \\ = g_t^2, \end{aligned}$$

we obtain

$$\mathbb{E}|u_{t+\tilde{t}}(x)|^2 \geq g_t^2 + g_t^2 \sum_{k=1}^{\infty} (\lambda l_{\sigma})^{2k} \int_{t-t/k}^t \int_{B_{\epsilon}^2} \int_{s_1-t/k}^{s_1} \int_{B_{\epsilon}^2} \cdots \int_{s_{k-1}-t/k}^{s_{k-1}} \int_{B_{\epsilon}^2} \\ \prod_{i=1}^k G_B(s_{i-1} - s_i, z_{i-1}, z_i) G_B(s_{i-1} - s_i, z'_{i-1}, z'_i) f(z_i, z'_i) dz_i dz'_i ds_i.$$

We now make a substitution and reduce the temporal region of integration to write

$$g_t^2 \sum_{k=1}^{\infty} (\lambda l_{\sigma})^{2k} \int_0^{t/k} \int_{B_{\epsilon}^2} \int_0^{t/k} \int_{B_{\epsilon}^2} \cdots \int_0^{t/k} \int_{B_{\epsilon}^2} \\ \prod_{i=1}^k G_B(s_i, z_{i-1}, z_i) G_B(s_i, z'_{i-1}, z'_i) f(z_i, z'_i) dz_i dz'_i ds_i.$$

We will further reduce the domain of integration so the function

$$\prod_{i=1}^k G_B(s_i, z_{i-1}, z_i) G_B(s_i, z'_{i-1}, z'_i) f(z_i, z'_i),$$

has the required lower bound. For $i = 0, \dots, k$, we set

$$z_i \in B(x, s_1^{\beta/\alpha}/2) \cap B(z_{i-1}, s_i^{\beta/\alpha})$$

and

$$z'_i \in B(x, s_1^{\beta/\alpha}/2) \cap B(z'_{i-1}, s_i^{\beta/\alpha}).$$

We therefore have $|z_i - z'_i| \leq s_1^{\beta/\alpha}$, $|z_i - z_{i-1}| \leq s_i^{\beta/\alpha}$ and $|z'_i - z'_{i-1}| \leq s_i^{\beta/\alpha}$. We

use the lower bound on the heat kernel from Lemma 2.5

$$\begin{aligned} & \prod_{i=1}^k G_B(s_i, z_{i-1}, z_i) G_B(s_i, z'_{i-1}, z'_i) f(z_i, z'_i) \\ & \geq \frac{c^k}{s_1^{k\gamma\beta/\alpha}} \prod_{i=1}^k \frac{1}{s_i^{2\beta d/\alpha}}, \end{aligned}$$

for some $c > 0$. We set $\mathcal{A}_i := B(x, s_1^{\beta/\alpha}/2) \cap B(z_{i-1}, s_i^{\beta/\alpha})$ and $\mathcal{A}'_i := B(x, s_1^{\beta/\alpha}/2) \cap B(z'_{i-1}, s_i^{\beta/\alpha})$. We will further choose that $s_i^{\beta/\alpha} \leq \frac{s_1^{\beta/\alpha}}{2}$ and note that $|\mathcal{A}_i| \geq c_1 s_i^{d\beta/\alpha}$ and $|\mathcal{A}'_i| \geq c_1 s_i^{d\beta/\alpha}$. We therefore have

$$\begin{aligned} & g_t^2 \sum_{k=1}^{\infty} (\lambda l_\sigma)^{2k} \int_0^{t/k} \int_{B_\epsilon^2} \int_0^{t/k} \int_{B_\epsilon^2} \cdots \int_0^{t/k} \int_{B_\epsilon^2} \\ & \prod_{i=1}^k G_B(s_i, z_{i-1}, z_i) G_B(s_i, z'_{i-1}, z'_i) f(z_i, z'_i) dz_i dz'_i ds_i \\ & \geq g_t^2 \sum_{k=1}^{\infty} (\lambda l_\sigma)^{2k} \int_0^{t/k} \int_{\mathcal{A}_1 \times \mathcal{A}'_1} \int_0^{t/k} \int_{\mathcal{A}_2 \times \mathcal{A}'_2} \cdots \int_0^{t/k} \int_{\mathcal{A}_k \times \mathcal{A}'_k} \\ & \frac{1}{s_1^{k\gamma\beta/\alpha}} \prod_{i=1}^k \frac{1}{s_i^{2\beta d/\alpha}} dz_i dz'_i ds_i \\ & \geq g_t^2 \sum_{k=1}^{\infty} (\lambda l_\sigma c_2)^{2k} \int_0^{t/k} \frac{1}{s_1^{k\gamma\beta/\alpha}} s_1^{k-1} ds_1 \\ & \geq g_t^2 \sum_{k=1}^{\infty} (\lambda l_\sigma c_3)^{2k} \left(\frac{t}{k} \right)^{k(1-\gamma\beta/\alpha)}. \end{aligned}$$

This completes the proof of proposition. \square

Proposition 4.2. *For any fixed $\epsilon > 0$, there exists a $t_0 > 0$ such that for all $0 < t \leq t_0$,*

$$\liminf_{\lambda \rightarrow \infty} \frac{\log \log \mathcal{I}_{\epsilon, t}}{\log \lambda} \geq \frac{2\alpha}{\alpha - \beta\gamma}.$$

Proof. Since t is strictly positive, we can use a substitution and rewrite the lower bound in Proposition 4.1 as

$$\begin{aligned} & \sum_{k=1}^{\infty} (\lambda l_\sigma c_3)^{2k} \left(\frac{t}{k} \right)^{k(1-\gamma\beta/\alpha)} \\ & = \sum_{k=1}^{\infty} \left(\frac{(\lambda l_\sigma c_3)^2 t^{(\alpha-\gamma\beta)/\alpha}}{k^{(\alpha-\gamma\beta)/\alpha}} \right)^k. \end{aligned} \tag{4.5}$$

Now from Lemma 2.10 with $\rho = (\alpha - \gamma\beta)/\alpha$ and $\theta = \lambda^2$ together with the previous proposition give the result. \square

Proof of Theorem 1.10. The previous propositions prove the theorem for all $0 < t \leq t_0$. Now we extend the result to all $t > 0$. For any $T, t > 0$,

$$\begin{aligned} \mathbb{E}|u_{t+T}(x)|^2 &\geq |(\mathcal{G}_B u)_{t+T}(x)|^2 + \lambda^2 l_\sigma^2 \int_0^{t+T} \int_{B^2} \\ &\quad G_B(T+t-s_1, x, z_1) G_B(T+t-s_1, x, z'_1) \mathbb{E}|u_{s_1}(z_1) u_{s_1}(z'_1)| f(z_1, z'_1) dz_1 dz'_1 ds_1. \end{aligned}$$

This leads to

$$\begin{aligned} \mathbb{E}|u_{t+T}(x)|^2 &\geq |(\mathcal{G}_B u)_{t+T}(x)|^2 + \lambda^2 l_\sigma^2 \int_0^t \int_{B^2} \\ &\quad G_B(t-s_1, x, z_1) G_B(t-s_1, x, z'_1) \mathbb{E}|u_{T+s_1}(z_1) u_{T+s_1}(z'_1)| f(z_1, z'_1) dz_1 dz'_1 ds_1. \end{aligned}$$

A similar argument used in the proof of Proposition 4.1 shows that

$$\begin{aligned} \mathbb{E}|u_{t+T}(x)|^2 &\geq |(\mathcal{G}_B u)_{t+T}(x)|^2 \\ &+ \sum_{k=1}^{\infty} (\lambda l_\sigma)^{2k} \int_0^t \int_{B^2} \int_0^{s_1} \int_{B^2} \cdots \int_0^{s_{k-1}} \int_{B^2} |(\mathcal{G}_B u)_{T+s_k}(z_k) (\mathcal{G}_B u)_{T+s_k}(z'_k)| \\ &\quad \prod_{i=1}^k G_B(s_{i-1}-s_i, z_{i-1}, z_i) G_B(s_{i-1}-s_i, z'_{i-1}, z'_i) f(z_i, z'_i) dz_i dz'_i ds_i. \end{aligned}$$

Similar ideas to those used in the proof of Proposition 4.1 combined with the proof of the proposition above show that for all $t \leq t_0$, we have

$$\liminf_{\lambda \rightarrow \infty} \frac{\log \log \mathbb{E}|u_{T+t}(x)|^2}{\log \lambda} \geq \frac{2\alpha}{\alpha - \beta\gamma},$$

for all $T > 0$ and whenever $x \in B(0, R - \epsilon)$. \square

Proof of Corollary 1.11. The proof of this corollary is exactly as that of Corollary 1.8 and it is omitted. \square

5 An extension

We can obtain results similar to the results in section 5 of [12]. We state one example, other examples can also be extended to the time fractional case. We

choose \mathcal{L} to be the generator of the relativistic stable process killed upon exiting the ball $B(0, R)$. So we are looking at the following equation

$$\begin{aligned}\partial_t^\beta u_t(x) &= m u_t(x) - (m^{2/\alpha} - \Delta)^{\alpha/2} u_t(x) + I_t^{1-\beta} [\lambda \sigma(u_t(x)) \dot{F}(t, x)], \quad x \in B(0, R), \\ u_t(x) &= 0, \quad x \in B(0, R)^c.\end{aligned}\tag{5.1}$$

Here m is a positive number. It is known that for any $\epsilon > 0$, there exists a $T_0 > 0$ such that for all $x, y \in B(0, R - \epsilon)$ and $t \leq T_0$ we have

$$p(t, x, y) \approx t^{-d/\alpha}$$

whenever $|x - y| \leq t^{1/\alpha}$. See, for instance, [7]. The constant involved in this inequality depends on m . We therefore have the same conclusion as the one in Theorem 1.10. So we have for all $x \in B(0, R - \epsilon)$

$$\lim_{\lambda \rightarrow \infty} \frac{\log \log \mathbb{E}|u_t(x)|^2}{\log \lambda} = \frac{2\alpha}{\alpha - \gamma\beta},$$

where u_t is the unique solution to (5.1).

References

- [1] B. Baeumer, M. Geissert, and M. Kovacs. Existence, uniqueness and regularity for a class of semilinear stochastic Volterra equations with multiplicative noise. Preprint.
- [2] B. Baeumer and M.M. Meerschaert. Stochastic solutions for fractional Cauchy problems, *Fractional Calculus Appl. Anal.* (2001) **4** 481–500.
- [3] J. Bertoin. *Lévy Processes*. Cambridge University Press, Cambridge (1996).
- [4] L. Boulanba, M. Eddahbi, and M. Mellouk. Fractional SPDEs driven by spatially correlated noise: existence of the solution and smoothness of its density. *Osaka J. Math.* Volume 47, Number 1 (2010), 41–65.
- [5] M. Caputo. Linear models of dissipation whose Q is almost frequency independent, Part II. *Geophys. J. R. Astr. Soc.* 13 (1967), 529–539.
- [6] Z.-Q. Chen, K.-H. Kim and P. Kim. Fractional time stochastic partial differential equations, *Stochastic Process Appl.* 125 (2015), 1470–1499.
- [7] Z.-Q. Chen, P. Kim and R. Song. Sharp heat kernel estimates for relativistic stable process in open sets. *Ann. Probab.*, 40(1), 2012.
- [8] L. Chen, Nonlinear stochastic time-fractional diffusion equations on \mathbb{R} : moments, Hölder regularity and intermittency. Preprint.

- [9] Z.-Q. Chen, M.M. Meerschaert and E. Nane. Space-time fractional diffusion on bounded domains. *Journal of Mathematical Analysis and Applications*. Volume 393, Issue 2, 15 September 2012, Pages 479488.
- [10] Dalang, Robert C.; Quer-Sardanyons, Lluís Stochastic integrals for spde's: a comparison. *Expo. Math.* 29 (2011), no. 1, 67109.
- [11] Giuseppe Da Prato and Jerzy Zabczyk, *Stochastic Equations in Infinite Dimensions*, *Encyclopedia of Mathematics and its Applications*, vol. 44, Cambridge University Press, Cambridge, 1992.
- [12] M. Foondun, K. Tian and W. Liu. On some properties of a class of fractional stochastic equations. Preprint available at [arxiv.org 1404.6791v1](https://arxiv.org/abs/1404.6791v1).
- [13] M. Foondun, W. Liu and M. Omaba, Moment bounds for a class of fractional stochastic equations. Preprint.
- [14] M. Foondun and D. Khoshnevisan. Intermittence and nonlinear parabolic stochastic partial differential equations, *Electron. J. Probab.* 14 (2009), no. 21, 548–568.
- [15] M. Foondun, D. Khoshnevisan and Eulalia Nualart. A local-time correspondence for stochastic partial differential equations, *Trans. Amer. Math. Soc.* 363 (2011), 2481–2515
- [16] M. Foondun and D. Khoshnevisan. On the stochastic heat equation with spatially-colored random forcing, *Trans. Amer. Math. Soc.* 365 (2013), 409–458
- [17] M. Foondun and E. Nane. Asymptotic properties of some space-time fractional stochastic equations. Preprint, 2015.
- [18] H. J. Haubold, A. M. Mathai and R. K. Saxena. Review Article: Mittag-Leffler functions and their applications, *Journal of Applied Mathematics*. Volume 2011 (2011) Article ID 298628, 51 pages
- [19] A. Karczewska. Convolution type stochastic Volterra equations, 101 pp., *Lecture Notes in Nonlinear Analysis* 10, Juliusz Schauder Center for Nonlinear Studies, Torun, 2007.
- [20] D. Khoshnevisan and K. Kim. Non-linear noise excitation and intermittency under high disorder, to appear in *Annals of probability*.
- [21] D. Khoshnevisan. *Analysis of stochastic partial differential equations*. CBMS Regional Conference Series in Mathematics, 119. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2014.
- [22] A.N. Kochubei, The Cauchy problem for evolution equations of fractional order, *Differential Equations*, 25 (1989) 967 – 974.

- [23] A. M. Mathai and H. J. Haubold, Special functions for applied scientists. Springer, 2007.
- [24] M.M. Meerschaert and H.P. Scheffler. Limit theorems for continuous time random walks with infinite mean waiting times. *J. Applied Probab.* **41** (2004) No. 3, 623–638.
- [25] M.M. Meerschaert, E. Nane and P. Vellaisamy. Fractional Cauchy problems on bounded domains. *Ann. Probab.* **37** (2009), 979–1007.
- [26] M.M. Meerschaert, E. Nane, and Y. Xiao. Fractal dimensions for continuous time random walk limits, *Statist. Probab. Lett.*, **83** (2013) 1083 – 1093
- [27] M.M. Meerschaert and P. Straka. Inverse stable subordinators. *Mathematical Modeling of Natural Phenomena*, Vol. 8 (2013), No. 2, pp. 1–16.
- [28] J. Mijena and E. Nane. Space time fractional stochastic partial differential equations. *Stochastic Process Appl.* **125** (2015), no. 9, 3301–3326
- [29] J. B. Mijena, and E. Nane. Intermittence and time fractional partial differential equations. Submitted. 2014.
- [30] T. Simon. Comparing Fréchet and positive stable laws. *Electron. J. Probab.* **19** (2014), no. 16, 1–25.
- [31] R.R. Nigmatullin. The realization of the generalized transfer in a medium with fractal geometry. *Phys. Status Solidi B.* **133** (1986) 425 – 430.
- [32] E. Orsingher and L. Beghin. Fractional diffusion equations and processes with randomly varying time. *Ann. Probab.* **37** (2009) 206 – 249.
- [33] S. Umarov and E. Saydamatov. A fractional analog of the Duhamel principle. *Fract. Calc. Appl. Anal.* **9** (2006), no. 1, 57 – 70
- [34] S.R. Umarov, and É. M. Saidamatov. Generalization of the Duhamel principle for fractional-order differential equations. (Russian) *Dokl. Akad. Nauk* **412** (2007), no. 4, 463–465; translation in *Dokl. Math.* **75** (2007), no. 1, 9496
- [35] S. Umarov. On fractional Duhamel’s principle and its applications. *J. Differential Equations* **252** (2012), no. 10, 5217 – 5234
- [36] John B. Walsh, An Introduction to Stochastic Partial Differential Equations, École d’été de Probabilités de Saint-Flour, XIV—1984, Lecture Notes in Math., vol. 1180, Springer, Berlin, 1986, pp. 265–439.
- [37] W. Wyss. The fractional diffusion equations. *J. Math. Phys.* **27** (1986) 2782 – 2785.
- [38] G. Zaslavsky. Fractional kinetic equation for Hamiltonian chaos. Chaotic advection, tracer dynamics and turbulent dispersion. *Phys. D* **76** (1994) 110–122.